ANALYSIS OF $L_2(s)$ AND TRIANGULAR DESIGNS

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1. INTRODUCTION

A general intrablock analysis of two class PBIB designs was given by Rao (1947) and Bose and Shimamoto (1952). Raghavarao (1962) gave an elegant analysis of $L_2(s)$ and group divisible designs, by using latent roots and latent vectors of their C-matrices. So far, no attempt seems to have been made to do the analysis of the $L_i(s)$ and triangular designs through the latent vectors and latent roots of their C-matrices, although such analysis for some higher class PBIB designs is available in literature. In this paper, we give the analysis of $L_i(s)$ and triangular designs with the help of latent vectors and latent roots of the C-matrices of these designs. The cumbersome expressions given by Rao (1947) and Bose and Shimamoto (1952) to estimate the treatment effects, can be avoided and the calculations for the estimates of treatment effects and variances of the estimates of elementary contrasts of treatment effects, can be very much simplified as is evident from the corresponding expressions discussed in sections 2 and 3 of this paper. An illustration showing application of the triangular designs as useful breeding experiment, is also given in section 3. For the definitions and notations of statistical terms used in this paper, we refer to Raghavarao (1971).

2. $L_i(s)$ DESIGNS

Let s^2 treatments of a connected $L_i(s)$ design with the parameters $v=s^2$, b,r,k, λ_1,λ_2 be given by an $s\times s$ array

$$(2.1)\qquad \begin{bmatrix} 11 & 12 & \dots & 1s \\ 21 & 22 & \dots & 2s \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ s1 & s2 & \dots & ss \end{bmatrix}$$

Let N be the incidence matrix of this $L_i(s)$ design. Then the latent roots θ_i of NN' with their multiplicities $\alpha_i (i=0, 1, 2)$ [*c.f.* Raghavarao (1971)] are as follows :

$$\theta_0 = rk, \ \alpha_0 = 1 ;$$

$$\theta_1 = r + (s-i)\lambda_1 - (s-i+1)\lambda_2, \ \alpha_1 = i(s-1) ;$$

$$\theta_2 = r - i\lambda_1 + (i-1)\lambda_2, \ \alpha_2 = (s-1)\ (s-i+1).$$

The latent roots ϕ_i of the C-matrix of this $L_i(s)$ design with their multiplicities α_i are

(2.3) $\phi_i = r - \theta_i / k \ (i=0, 1, 2).$

Let

(2.4) $Y_{jkl} = m + t_{jk} + \beta_e + e_{jkl},$

$j, k = 1, 2, \ldots, s ;$

$l = 1, 2, \ldots, b ;$

Y_{jkl} being the yield of the plot of the lth block to which the jkth treatment is applied ; m being the general mean ; t_{jk} being the effect of the jkth treatment and β_e being the effect of the lth block. m, t_{jk}'s, β_e's are assumed to be the fixed effects. e_{jkl}'s are independent and normal random variates with expectation 0 and variance σ^2.

Let

(2.5) $$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}, \ p = t, Q, \hat{t}$$

be the $s \times s$ arrays for the treatment effects, adjusted treatment totals and least square estimates of treatment effects, respectively.

Let

(2.6) $$R_j^p = \sum_{k=1}^{s} p_{jk}, \ C_k^p = \sum_{j=1}^{s} p_{jk},$$

$$p = (p_{11}, p_{12}, \ldots p_{1s}, \ldots p_{s1}, p_{s2}, \ldots p_{ss})', \ p = t, Q, \hat{t}.$$

Further, let $(i-2)$ mutually orthogonal latin squares (MOLS) exist. Let $M_1^{(l)t}, \ldots, M_s^{(l)t}$ be the totals of the treatment effects obtained by superimposing the lth latin square on the $s \times s$ array

(2.5), $M_j(l)_t$ representing the total of t_{jk}'s corresponding to the jth letter of the lth latin square $[j=1, 2, \ldots, s; l=1, 2, \ldots, (i-2)]$ Let $M_1^{(l)Q}, \ldots, M_s^{(l)Q} (l=1, 2, \ldots, i-2)$ represent the corresponding total of Q_{ij}'s—the adjusted treatment totals. Then $i(s-1)$ orthonor mal latent vectors $\underline{x}_{Sj}$ $(S=R, C, M^{(1)}, \ldots, M^{(i-2)}; j=1, 2, \ldots, s-1)$ of the latent root ϕ_1 of the C-matrix of the given $L_i(s)$ design, are given by

$$(2.7) \qquad \underline{x}'_{Sj} \underline{t} = [\sum_{m=1}^{j} S_m{}^t - j\, S^t{}_{j+1}] - [j(j+1)s]^{1/2}.$$

Let $(s-1)(s-i+1)$ orthonormal latent vectors corresponding to the latent root ϕ_2 of the C-matrix be

$$(2.8) \qquad \underline{y}_{mj} (m=1, 2, \ldots, s-i+1; j=1, 2, \ldots, s-1).$$

Clearly $\underline{y}_{mj}$'s will be orthogonal to $\underline{x}_{Sj}$'s. Let

$$(2.9) \qquad A_1 = \Sigma\Sigma\, \underline{x}_{Sj} \underline{x}'_{Sj}$$

Then following Raghavarao (1962), a solution of the reduced normal equations

$$(2.10) \qquad C \hat{\underline{t}} = Q$$

will be

$$(2.11) \qquad \hat{\underline{t}} = [(1/\phi_1)A_1 + (1/\phi_2)(I_v - A_1)]\, \underline{Q},$$

which on simplification becomes

$$(2.12) \qquad \hat{t}_{jk} = Q_{jk}/\phi_2 + (1/s)(1/\phi_1 - 1/\phi_2)[R_j^Q + C_k^Q + M_{p1}{}^{(1)Q} + M_{p2}{}^{(2)Q} + \ldots + M_{pi-2}{}^{(i-2)Q}],\ j, k = 1, 2, \ldots, s,$$

where $p_l (l=1, 2, \ldots, i-2)$ is the letter of the lth latin square corresponding to the jkth symbol of $L_i(s)$ association scheme given by (2.1), when the later is superimposed on the former.

As an illustration, let

$$(2.13)\qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}$$

be two MOLS of order 4. For a $L_4(4)$ design

$$(2.14)\quad \hat{t}_{23} = Q_{23}/\phi_2 + (1/4)(1/\phi_1 - 1/\phi_2)(R_2^Q + C_3^Q + M_4^{(1)Q} + M_3^{(2)Q}).$$

Sum of squares due to treatments eliminating blocks will be

$$(2.15)\quad (1/\phi_2)\Sigma\Sigma Q^2_{ij} + (1/s)(1/\phi_1 - 1/\phi_2)[\Sigma(M_j^{(1)}Q)^2 + \ldots\ldots +$$
$$\Sigma(M_j^{(i-2)}Q)^2 + \Sigma(R_i^Q)^2 + \Sigma(C_j^Q)^2].$$

The variances of elementary contrasts are given by

$$(2.16)\quad V(\hat{t}_{ij} - \hat{t}_{kl}) = 2\sigma^2[(1/\phi_2) + (1/s)(1/\phi_1 - 1/\phi_2)(i-1)]$$

or
$$2\sigma^2[(1/\phi_2) + (1/s)(1/\phi_1 - 1/\phi_2)i]$$

according as ijth, klth treatments are 1st or 2nd associates. The average variance is

$$(2.17)\quad 2\sigma^2[i(s-1)/\phi_1 + (s-1)(s-i+1)\phi_2]/(v-1)$$

as it ought to be.

3. Triangular Designs

Let the $s(s-1)/2$ treatments of a connected triangular design with the parameters v, b, r, k, λ_1, λ_2 be represented by an $s \times s$ array

$$(3.1)\qquad \begin{bmatrix} * & 12 & 13 & . & . & . & 1s \\ 21 & * & 23 & . & . & . & 2s \\ 31 & 32 & * & . & . & . & 3s \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & * & (s-1)s \\ s1 & s2 & s3 & . & . & s(s-1) & * \end{bmatrix},$$

with $ij=ji(i\neq j)$, $i, j=1, 2, 3, \ldots, s$.

Let N be the incidence matrix of this triangular design. Then the latent roots θ_i of NN' with multiplicities $\alpha_i(i=0, 1, 2)$ [*c.f.* Raghavarao (1971)] are

$$\theta_0=rk,\ \alpha_0=1;\ \theta_1=r+(s-4)\lambda_1-(s-3)\lambda_2,\ \alpha_1=s-1;$$

(3.2) $$\theta_2=r-2\lambda_1+\lambda_2,\ \alpha_2=s(s-3)/2.$$

The latent roots ϕ_i of the C-matrix of the given triangular design will be given by

(3.3) $$\phi_i=r-\theta_{i/k}$$

with their respective multiplicities $\alpha_i(i=0, 1, 2)$.

Let

(3.4) $$Y_{ijk}=m+t_{ij}+\beta_k+e_{ijk},\ i, j=1, 2, \ldots, s;\ ij=ji;\ i\neq j,$$

Y_{ijk} being the yield of the plot of the kth block to which ijth treatment is applied; m being the general mean; t_{ij} being the effect of the ijth treatment and β_k is the effect of the kth block. m, t_{ij}'s and β_k's are assumed to be the fixed effects. e_{ijk}'s are normal and independent variates with expectation 0 and variance σ^2. This model is known as the fixed model or model I of Eisenhart (1947).

Let

(3.5) $$\begin{bmatrix} * & p_{12} & p_{13} & \cdot & \cdot & \cdot & p_{1s} \\ p_{21} & * & p_{23} & \cdot & \cdot & \cdot & p_{2s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{s-1,1} & \cdot & \cdot & \cdot & \cdot & * & p_{s-1,s} \\ p_{s1} & p_{s2} & p_{s3} & \cdot & \cdot & p_{s,s-1} & * \end{bmatrix},\ p=t,\ \hat{t},\ Q$$

be the $s\times s$ arrays of treatment effects t_{ij}'s, least square estimates of t_{ij}'s and adjusted treatment totals Q_{ij}'s respectively. Let

$$\underline{p}=(p_{12}, \ldots, p_{1s}, p_{23}, \ldots, p_{2s}, \ldots, p_{s-1,s})',$$

(3.6) $$R_i^p=\sum_j p_{ij},\ C_j^p=\sum_i p_{ij};\ p=t,\ \hat{t},\ Q$$

where $ij=ji$ and $i \neq j$. Then $(s-1)$ orthonormal latent vectors $\underline{x}_i (i=2, 3, \ldots, s)$ corresponding to the latent root ϕ_1 of the C-matrix, will be given by

(3.7) $\underline{x}'_i \underline{t} = [\sum_{m=1}^{i-1} R_m^t - (i-1) R_i^t] \div [(i-1)i(s-2)]^{1/2}, \; i=2, 3, \ldots, s$

where $(ij=ji, \; i \neq j)$. Let $s(s-3)/2$ orthonormal latent vectors corresponding to the latent root ϕ_2 of the C-matrix be

(3.8) $\underline{y}_j (j=1, 2, \ldots, s(s-3)/2)$.

Let

(3.9) $A_1 = \Sigma \underline{x}_i \underline{x}'_i$.

A solution of the reduced normal equations

(3.10) $C \hat{\underline{t}} = \underline{Q}$

will be

(3.11) $\hat{\underline{t}} = [(1/\phi_1) A_1 + (1/\phi_2)(I_v - A_1)] \underline{Q}$,

which on simplification becomes

(3.12) $\hat{t}_{ij} = Q_{ij}/\phi_2 + (R_i^Q + R_j^Q)(1/\phi_1 - 1/\phi_2)/(s-2)$.

$i, j = 1, 2, \ldots, s; \; i \neq j; \; ij = ji$.

Sum of squares due to treatments eliminating blocks will be

(3.13) $\Sigma\Sigma Q^2_{ij}/\phi_2 + (1/\phi_1 - 1/\phi_2) \Sigma (R_i^Q)^2/(s-2), \; i<j$.

Variances of elementary contrasts will be given by

(3.14) $V(\hat{t}_{ij} - \hat{t}_{kl}) = 2\sigma^2[(1/\phi_2) + (1/\phi_1 - 1/\phi_2)/(s-2)]$

or $2\sigma^2[(1/\phi_2) + 2(1/\phi_1 - 1/\phi_2)/(s-2)]$

according as ijth and klth treatments are 1st or 2nd associates. The average variance of elementary contrasts is

(3.15) $2\sigma^2[(s-1)/\phi_1 + s(s-3)/2\phi_2]/(v-1)$

as it ought to be.

For the definitions of various breeding terms, we refer to Sprague and Tatum (1942) or Griffing (1956).

Let

(3.16) $t_{ij} = g_i + g_j + s_{ij}$,

g_i being the *g.c.a.* effect of ith line and s_{ij} being the *s.c.a.* effect due to the cross of ith and jth lines. We assume

(3.17) $\sum_i g_i=0$; $\sum_i s_{ij}=0$, $\forall j$.

We can easily see that

(3.18) $g_i=R_i^t/(s-2)$, $s_{ij}=t_{ij}-(R_i^t+R_j^t)/(s-2)$.

The relations (3.18) imply that the $(s-1)$ orthogonal latent vectors $\underline{x}_i(i=2, 3, ..., s)$ for the latent root ϕ_1 of the C-matrix are the $(s-1)$ *g.c.a.* effects comparisons. Thus other $s(s-3)/2$ orthogonal comparisons $\underline{y}_j(j=1, 2, ..., s(s-3)/2)$ will be the *s.c.a.* effects comparisons. Further it can be easily seen that the sum of squares due to *g.c.a.* effects eliminating blocks will be

(3.19) $\Sigma(R_i^Q)^2/(s-2)\phi_1$.

The anova table is given in Table 3.1.

The estimates of *g.c.a.* and *s.c.a.* effects, their variances and variances of the estimate of their elementary contrasts are

$$\hat{g}_i=(1/(s-2)\phi_1)R_i^Q,\ \hat{s}_{ij}=[R_{ij}-(R_i^Q+R_j^Q)/(s-2)]/\phi_2,$$

$$V(\hat{g}_i)=\sigma^2(s-1)/s(s-2)\phi_1,\ V(\hat{s}_{ij})=\sigma^2(s-3)/(s-1)\phi_2,$$

(3.20) $$V(\hat{g}_i-\hat{g}_j)=2\sigma^2/(s-2)\phi_1,$$

$$V(\hat{s}_{ij}-\hat{s}_{ik})=2\sigma^2(s-3)/(s-2)\phi_2(j\neq k),$$

$$V(\hat{s}_{ij}-\hat{s}_{kl})=2\sigma^2(s-4)/(s-2)\phi_2(i=j,k,l;\ j\neq k,\ l;\ k\neq l).$$

Let us again consider equations (3.4) and (3.16). Let m, β_e's be the fixed effects and let g_i's, s_{ij}'s and e_{ijk}'s be normally and independently distributed with expectations zero and variances σ_g^2, σ_s^2 and σ^2. Let these random variables be pairwise uncorrelated. The system given by (3.4) and (3.16) with these assumptions is called the mixed model [*see* Searle (1971) p. 381]. For the fixed effects model significances of g_i's and s_{ij}'s are tested by calculating the ratios M_g/M_e and M_s/M_e whereas for testing σ_s^2 the ratio M_s/M_e is used and for testing σ_g^2 (if $\sigma_s^2\neq 0$), Scheffe's (1959, p. 247-48) approximate test is made use of otherwise M_s and M_e can be pooled and σ_g^2 is tested in the usual way. The expectations of mean square for the two models is given in the anova table 3.1.

Illustration 3.1. Let us consider the triangular design with the parameters $v=6$, $b=4$, $r=2$, $k=3$, $\lambda_1=1$, $\lambda_2=0$ and with the

TABLE 3.1 *Anova Table*

Source	*d.f.*	*S.S.*	*M.S.*	*E(M.S) Model I*	*E(M.S) Mixed Model*
Blocks ignoring treatments	$b-1$	$(1/k)\Sigma B_j^2 - C.F.$	—	—	—
g.c.a. eliminating blocks	$s-1$	$\sum_{i=1}^{s} (R_i^Q)^2/(s-2)\phi_1$	M_g	$\sigma^2+(s-2)\phi_1\Sigma g^2{}_i/(s-1)$	$\sigma^2+(s-2)\phi_1\sigma_g^2+\phi_1\sigma_s^2$
s.c.a. eliminating blocks	$s(s-3)/2$	$[\sum\sum_{i<j} Q^2{}_{ij} - \sum_{i=1}^{s} (R_i^Q)^2/(s-2)]/\phi_2$	M_s	$\sigma^2+2\phi_2 \sum\sum_{i<j} s_{ij}^2/s(s-3)$	$\sigma^2+\phi_2\sigma_s^2$
Error	$vr-v-b+1$	By subtraction	M_e	σ^2	σ^2
Total	$vr-1$	$\Sigma\Sigma\Sigma y^2{}_{ijk} - C.F.$	—	—	—

triangular association scheme :

$$\text{(3.21)} \qquad \begin{bmatrix} * & 12 & 13 & 14 \\ 12 & * & 23 & 24 \\ 13 & 23 & * & 34 \\ 14 & 24 & 34 & * \end{bmatrix}.$$

Let us assume a fixed effect model. Let the yields (given within brackets) of the $6F_1$'s be

$$\text{(3.22)} \qquad \begin{matrix} [12(7), & 13(10), & 14(11)], & [12(9), & 23(14), & 24(16)], \\ [13(11), & 23(13), & 34(17)], & [14(13), & 24(18), & 34(20)]. \end{matrix}$$

The data is factitious. Then the Q_{ij} matrix will be

$$\text{(3.23)} \qquad \begin{bmatrix} * & -6.33 & -2.00 & -2.33 \\ -6.33 & * & 0.33 & 4.00 \\ -2.00 & 0.33 & * & 6.33 \\ -2.33 & 4.00 & 6.33 & * \end{bmatrix}.$$

The anova table is as given below :

ANOVA TABLE

Source	*d.f.*	*S.S.*	*M.S.*	*F-ratio*
Blocks ignoring treatments	3	88.92	—	—
g.c.a. eliminating blocks	3	76.26	25.42	70.8**
s.c.a eliminating blocks	2	2.00	1.00	2.78
Error	3	1.08	0.36	—
Total	11	168.25		

The *g.c.a.* effects are significantly different at 1 p.c. level of significance. Their estimates are

$$\text{(3.24)} \qquad \hat{g}_1 = -4.00, \quad \hat{g}_2 = -0.75, \quad \hat{g}_3 = 1.75, \quad \hat{g}_4 = 3.00.$$

C.D. of these estimates at 5 p.c. level of significance is 1.65.

The latent roots of NN' where N is the incidence matrix of a PBIB design with m classes, play an important role in determining the relative loss of information for the partially confounded sets of degrees of freedom. Shah (1958) proved that the relative loss of information on each of α_i degrees of freedom, was

(3.25) $$\theta_i/rk$$

where θ_i is the latent root of NN' with multiplicity $\alpha_i (i=1, 2, .., m)$. For the series of triangular designs [*see* Shrikhande (1965) or Raghavarao (1970)]

(3.26) $v=(2n-1)n,\ b=(2n-1)(2n-3),\ r=2n-3,\ k=n,\ \lambda_1=0,\ \lambda_2=1$

the relative loss of information on each of the *g.c.a.* degrees of freedom will be zero and on each of *s.c.a.* degrees of freedom will be $2(s-1)/(2n-3)n$. The relative loss of information on *s.c.a.* degrees of freedom for the triangular design [*see* Raghavarao (1971)].

(3.27) $v=s(s-1)/2,\ b=s,\ r=2,\ k=s-1,\ \lambda_1=1,\ \lambda_2=0$

is zero and on each of *g.c.a.* degrees of freedom is $(s-2)/2(s-1)$. The design (3.27) always exists whereas the existence of the series of designs (3.26) for all values of n has not been established, so far.

Summary

The paper contains analysis of $L_i(s)$ and Triangular Designs through the latent vectors and latent roots of their C-matrices. Application of triangular designs as diallel cross experiments-method (4) of Griffing (1956) involving s inbred lines is also given therein.

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